Supplement 1

Text S1 Deviation of the equilibrium of the infection number distribution

For making the calculation easy, we first use the following equation instead of Eqn.1, with general functions $a(n)$ and $r(n)$ for new infection rate and partial recovery rate, respectively:

$$\frac{dH_n}{dt} = a(n-1)H_{n-1}F - a(n)H_nF + (n+1)r(n+1)H_{n+1} - nr(n)H_n \ .$$

With timescale separation (setting $dH_n/dt = 0$ and treating F as constant), we must solve the following equation to obtain the equilibrium infection number distribution:

$$0 = a(n-1)H_{n-1}F - a(n)H_nF + (n+1)r(n+1)H_{n+1} - nr(n)H_n \ .$$

It gives:

$$H_{n+1} = \frac{-a(n-1)H_{n-1}F + \{a(n)F + nr(n)\}H_n}{(n+1)\cdot r(n+1)}.$$

By converting it to the probability unit and changing the index from n+1 to n, we obtain:

$$P_n \equiv \frac{H_n}{\sum H_k} = \frac{-a(n-2)FP_{n-2} + [(n-1)r(n-1) + r(n-1)F]P_{n-1}}{nd(n)}.$$

For example, we can calculate:

$$P_1 = \frac{a(0)F}{r(1)}P_0, \quad P_2 = \frac{-a(0)P_0F + [r(1) + r(1)F]P_1}{2r(2)} = \frac{a(1)a(0)F^2}{2r(2)r(1)}P_0,$$

and $$P_3 = \frac{-a(1)P_1F + [2r(2) + r(2)F]P_2}{3r(3)} = \frac{1}{3!}\frac{a(2)a(1)a(0)F^3}{r(3)r(2)r(1)}P_0$$

Then, using the inductive method, we obtain the following probability distribution:

$$P_n = \frac{1}{n!} F^n \prod_{k=1}^{n} \frac{a(k-1)}{r(k)} P_0 \quad n \geq 1. \ (4)$$

Let:

$$\begin{aligned} a_0 &= a(0) \\ r_0 &= r(0) \qquad (5) \\ \lambda &= \frac{a_0}{r_0} F \equiv \lambda(F). \end{aligned}$$

Then, Eqn. 4 is equivalent to:

$$P_n = \frac{\lambda^n}{n!} P_0 \prod_{k=1}^{n} \frac{a(k-1)/a_0}{r(k)/r_0} \quad n \geq 1. \quad (6)$$

With the constraint $1 = \sum_{n=0}^{\infty} P_n$, we obtain:

$$P_0 = \frac{1}{1 + \sum_{n=1}^{\infty} \left[\frac{\lambda^n}{n!} \prod_{k=1}^{n} \frac{a(k-1)/a_0}{r(k)/r_0} \right]} \equiv P_0(\lambda). \quad (7)$$

Of note, when the function $a(k)$ and $r(k)$ are constant as a_0 and r_0, respectively, the denominator of Eqn. 7 becomes the Taylor series of the exponential function of λ. Therefore, Eqn. 7 becomes exp(–□) and Eqn. 6 can be simplified as the Poisson distribution.

Then we used the specific functions $a(n) = a_0 \exp(a_1 n + a_2 n^2)$ and $r(n) = r_0 \exp(r_1 n + r_2 n^2)$ for new infection and partial recovery, respectively. Here, a_0 and r_0 are density-independent coefficients and a_1 a_2, d_1, and d_2 are density-dependent ones. In this case, we have,

$$P_n = \frac{\lambda^n}{n!} P_0 \prod_{k=1}^{n} \frac{\exp\left(a_1(k-1) + a_2(k-1)^2\right)}{\exp\left(r_1 k + r_2 k^2\right)} \qquad n \geq 1$$

$$= \frac{1}{n!}\left(\frac{a_0}{r_0}F\right)^n P_0 \exp \sum_{k=1}^{n} \left[(a_2 - r_2)k^2 + (a_1 - 2a_2 - r_1)k + (a_2 - a_1)\right]$$

$$= \frac{1}{n!}\left(\frac{a_0}{r_0}F\right)^n P_0 \exp\left[\frac{a_1 - r_2}{3}n^3 + \frac{a_1 - a_2 - r_1 - r_2}{2}n^2 + \left(-\frac{a_1 + r_1}{2} + \frac{a_2 - r_2}{6}\right)n\right]$$

$$= P_0 \frac{1}{n!}\left(\frac{a_0}{r_0}\exp\left(-\frac{a_1 + r_1}{2} + \frac{a_2 - r_2}{6}\right)F\right)^n \exp\left[\frac{a_1 - r_2}{3}n^3 + \frac{a_1 - a_2 - r_1 - r_2}{2}n^2\right],$$

which gives Eqn.2.

Fig. S1 Yoneya et al.

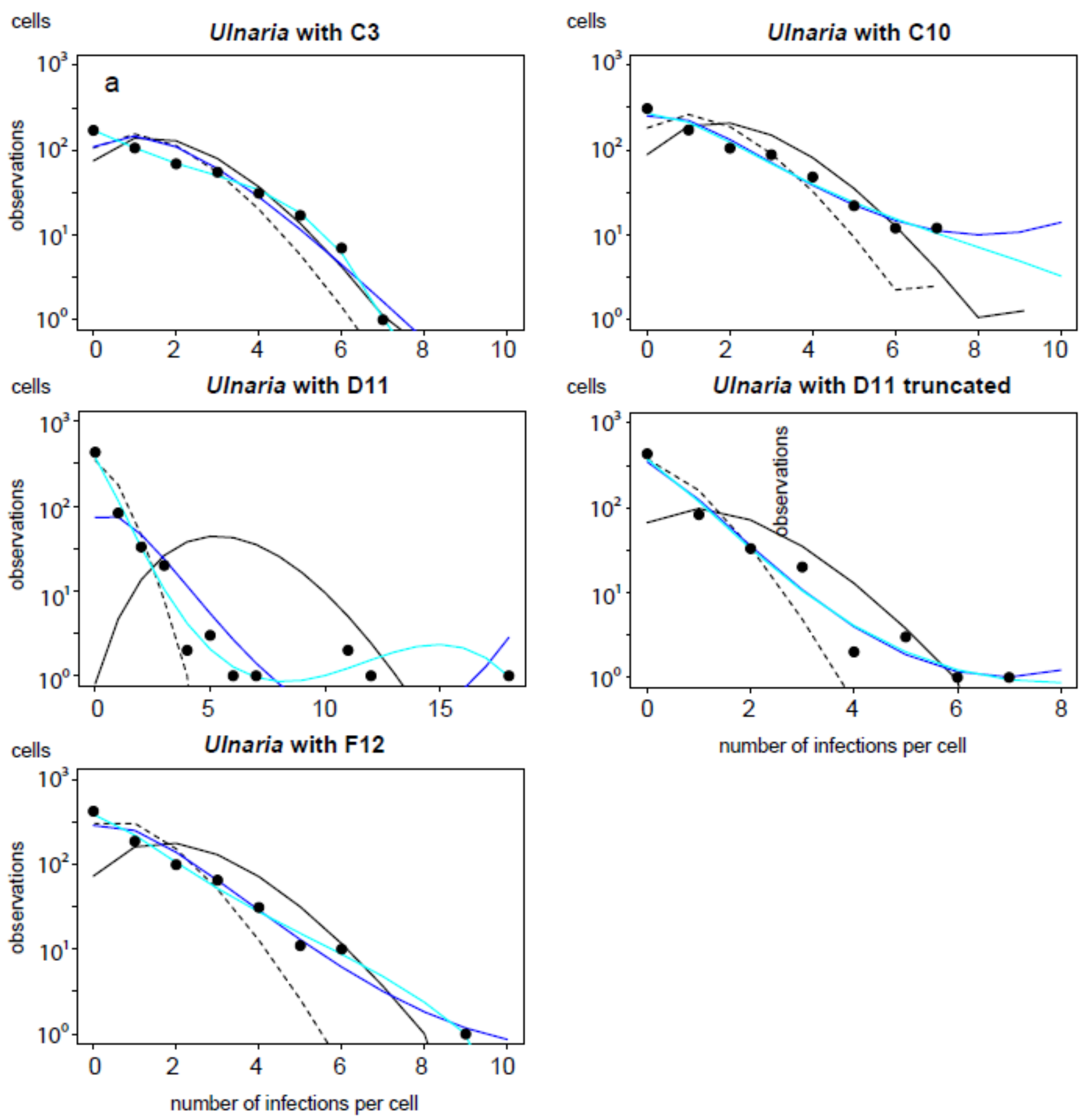

Fig. S1 Model fitting to the observed infection number distribution. The dashed and solid black line represent the Poisson fitting with a single parameter λ and 1st order model fitting with two parameters (b_0, b_1 in Eqn.3), respectively. Blue and light-blue lines represent the 2nd order (b_2 >0, b_3 = 0) and 3rd order (b_3 > 0) model fitting, respectively. For the *Ulnaria* system with D11, one may argue that the accuracy for counting large infections number (> 10) was low, then we also fit the models to the data when the empirical results with n > 10 were truncated.